Iwasawa theory and congruences for the symmetric square of a modular form

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Joint with Ramdorai Sujatha and Vinayak Vatsal.

Selmer groups associated with elliptic curves

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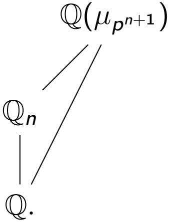
- For $n \in \mathbb{Z}_{\geq 1}$, let $\mathbb{Q}_n$ be the subfield of $\mathbb{Q}(\mu_{p^{n+1}})$ such that $\mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q}) \simeq \mathbb{Z}/p^n$ as depicted



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- The Selmer group over $\mathbb{Q}_{\mathrm{cyc}}$ is taken to be the limit

$$\mathrm{Sel}_{p^\infty}(E/\mathbb{Q}_{\mathrm{cyc}}) := \varinjlim_{n \to \infty} \mathrm{Sel}_{p^\infty}(E/\mathbb{Q}_n).$$

Congruent modular forms

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- Let
$$g_1 = \sum_{n=1}^{\infty} a(n, g_1) q^n \text{ and } g_2 = \sum_{n=1}^{\infty} a(n, g_2) q^n$$
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- Let L be the number field generated by the Fourier coefficients of g_1 and g_2 and $\mathfrak{p}$ a prime of $\mathcal{O}_L$ above p.
- We say that g_1 and g_2 are $\mathfrak{p}$-congruent if
$$a(n, g_1) \equiv a(n, g_2) \mod \mathfrak{p}$$
for all n coprime to $M_1 M_2 p$.

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- Choose a Galois stable $\mathcal{O}$-lattice $T_i \subset V_i$ and let

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- Assume throughout that $\bar{\rho}_i$ is absolutely irreducible for $i = 1, 2$, and thus the residual representations are isomorphic.

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- Let $\mathbb{T}_i$ be the underlying Galois stable $\mathcal{O}$-lattice, on which $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ acts via r_i and $\mathbb{A}_i := \mathbb{T}_i \otimes \mathbb{Q}_p/\mathbb{Z}_p$ the associated p-divisible groups.

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- Let Σ be the set of primes dividing $M_1 M_2 p$, and $\mathbb{Q}_\Sigma \subset \bar{\mathbb{Q}}$ the maximal extension of $\mathbb{Q}$ unramified at all primes $\ell \notin \Sigma$.
- The p-primary Selmer group $\text{Sel}(\mathbb{A}_i/\mathbb{Q}_{\text{cyc}})$ is defined as the kernel of the following restriction map

$$H^1(\mathbb{Q}_\Sigma/\mathbb{Q}_{\text{cyc}}, \mathbb{A}_i) \to \bigoplus_{\ell \in \Sigma} \mathcal{H}_\ell(\mathbb{A}_i/\mathbb{Q}_{\text{cyc}}).$$

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- The Iwasawa algebra Λ is defined as the following inverse limit $\Lambda := \varprojlim_n \mathbb{Z}_p[\mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})]$, and is isomorphic to the formal power series ring $\mathbb{Z}_p[[T]]$.

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- Loeffler and Zerbes showed that the Selmer group $\mathrm{Sel}(\mathbb{A}_i/\mathbb{Q}_{\mathrm{cyc}})$ is a cotorsion Λ-module.
- Let M be a cofinitely generated, cotorsion $\mathbb{Z}_p[[T]]$-module and $M^\vee$ its Pontryagin-dual.

- By the structure theory of $\mathbb{Z}_p[[T]]$ modules, up to a pseudoisomorphism, $M^\vee$ decomposes into cyclic-modules:

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- Set $\mu(M) = \sum \mu_j$ and $\lambda(M) = \sum \deg f_j$.
- The characteristic element is the product $\mathrm{char}_M := p^\mu \times \prod f_j$.

Conjecture (Iwasawa main conjecture)

Let f be a p-ordinary Hecke eigencuspform of weight $k \geq 2$,

$$\mathrm{char}(\mathrm{Sel}_{p^\infty}(\mathbb{A}_f/\mathbb{Q}_{\mathrm{cyc}})) = U(T) \cdot L_p(\mathrm{Sym}^2(f), T),$$

for some unit $U(T) \in \Lambda^\times$.

Algebraic & analytic μ and λ-invariants

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- For $i = 1, 2$, let μ_i^{an} and λ_i^{an} be the Iwasawa-invariants of $L_p(\text{Sym}^2(g_i), T)$. In other words,

$$L_p(\text{Sym}^2(g_i), T) = p^\mu a(T)u(T),$$

where $\mu = \mu_i^{\text{an}}$, $a(T)$ is a distinguished polynomial of degree λ_i^{an} and $u(T)$ is a unit in Λ.

Main result

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Theorem (R, R. Sujatha, V. Vatsal)

Let g_1 and g_2 be non-CM, p-ordinary, $\mathfrak{p}$-congruent newcuspforms of weight $k \geq 2$ and trivial nebentype character. Let M_i be the level of g_i and Σ_0 be the set of primes $\ell \neq p$ such that $\ell | M_1 M_2$. Assume that $p > \max\{k-2, 3\}$. Then, for $* \in \{\text{alg}, \text{an}\}$, we have that $\mu_1^* = 0 \Leftrightarrow \mu_2^* = 0$. Furthermore, if $\mu_1^* = 0$ (or equivalently, $\mu_2^* = 0$), then,

$$\lambda_1^* - \lambda_2^* = \sum_{\ell \in \Sigma_0} \left(\sigma_2^{(\ell)} - \sigma_1^{(\ell)} \right).$$

In the case when $5 \leq p \leq k-2$, the result is conditional, provided a certain version of Ihara's lemma holds.

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- The imprimitive Selmer group $\mathrm{Sel}^{\Sigma_0}(\mathbb{A}_i/\mathbb{Q}_{\mathrm{cyc}})$ is the kernel of the map
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- These Selmer groups fit into a short exact sequence
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- Let $\sigma_i^{(\ell)}$ be the $\mathbb{Z}_p$-corank of $\mathcal{H}_\ell(\mathbb{A}_i/\mathbb{Q}_{\mathrm{cyc}})$, we find that
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- The imprimitive λ-invariants are equal $\lambda_1^{\mathrm{alg},\Sigma_0} = \lambda_2^{\mathrm{alg},\Sigma_0}$. This translates into the following relationship between primitive λ-invariants
$$\lambda_1^{\mathrm{alg}} - \lambda_2^{\mathrm{alg}} = \sum_{\ell \in \Sigma_0} \left(\sigma_2^{(\ell)} - \sigma_1^{(\ell)}\right).$$

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- The Petersson product of modular forms f_1, f_2 of weight k on $\Gamma_0(M)$ (at least one of which is cuspidal) to be as follows:

$$\langle f_1, f_2\rangle_M = \int_{B(M)} f_1(z)\overline{f_2(z)}y^{k-2}dxdy$$

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- One sees from the definition that $\{\cdot, \cdot\}$ is $\mathbb{C}$-linear in both variables, and that it satisfies $\{v|t, w\}_N = \{v, w|t\}_N$, for any Hecke operator t.

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- Let $g = g_i$, the L-function $L(s, g) = \sum a(n, g)n^{-s}$ of g has the formal Euler product expansion

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- The naive symmetric square L-function is as follows:

$$D_f(\chi, s)$$

$$= \left(\prod_{\ell \nmid N}(1 - \chi(\ell)\alpha_\ell\beta_\ell q^{-s})(1 - \chi(\ell)\beta_\ell^2 q^{-s})(1 - \chi(\ell)\alpha_\ell^2 q^{-s})\right)^{-1}.$$

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- The following formula is due to Shimura

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- For an odd integer n in the range $1 \leq n \leq k-1$, set $H_{\overline{\chi}}(n) := \theta_{\overline{\chi}}(z)\Phi(z, \overline{\chi}, n)$. Sturm has shown that $H_{\overline{\chi}}(n)$ is a nearly holomorphic modular form of level N_χ, weight k, and trivial character.

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- Then the eigenform f determines a ring homomorphism $\mathbb{T} \to \mathcal{O}$, sending a Hecke-operator $T \in \mathbb{T}$ to the T-eigenvalue of f. Set $\mathcal{P}$ to denote the kernel of this homorphism. There is a unique maximal ideal $\mathfrak{m}$ of $\mathbb{T}$ that contains $\mathcal{P}$ and the maximal ideal of $\mathcal{O}$.

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- There is a canonical duality of $\mathbb{T}_\mathfrak{m}$-modules between $S_k(N, \mathcal{O})_\mathfrak{m}$ and $\mathbb{T}_\mathfrak{m}$ defined by the form $S_k(N, \mathcal{O})_\mathfrak{m} \times \mathbb{T}_\mathfrak{m} \to \mathcal{O}$ mapping $(s, t) \mapsto a(1, s|t) \in \mathcal{O}$.

- Since $\mathbb{T}_{\mathfrak{m}}$ is Gorenstein, there is an isomorphism of $\mathbb{T}_{\mathfrak{m}}$-modules

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- Define the period of f at level N as $\Omega_N = \frac{\{f,f\}_N}{(f,f)_N}$.

- Since $\mathbb{T}_{\mathfrak{m}}$ is Gorenstein, there is an isomorphism of $\mathbb{T}_{\mathfrak{m}}$-modules

$$\iota : \mathbb{T}_{\mathfrak{m}} \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}}\left(\mathbb{T}_{\mathfrak{m}}, \mathcal{O}\right) \xrightarrow{\sim} S_k(N, \mathcal{O}).$$

- Define an algebraic pairing $(\cdot, \cdot)$

$$(\cdot, \cdot)_N : S_k(N, \mathcal{O}) \times S_k(N, \mathcal{O}) \to \mathcal{O}$$

by $(v, w) := a\left(1, v_{|\iota(w)}\right)$.
- Define the period of f at level N as $\Omega_N = \frac{\{f,f\}_N}{(f,f)_N}$.
- In order to relate the p-adic L-function of f to the imprimitive p-adic L-function associated to g, we show that $\Omega_N = u\Omega_M$ for a unit $u \in \mathcal{O}$.

- Recall that $H_\chi(n) = \theta_\chi(z)\Phi(z, \chi, n)$.

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- Set

$$\widetilde{H}_\chi(n) = \frac{\Gamma((n+1)/2)}{\pi^{(1+n)/2}} p^{m_\chi(3-2k+2n)/2} \cdot \frac{\sqrt{Cp^{m_\chi}}}{G(\chi)} \cdot H_\chi(n) \circ W_{N_\chi},$$

it follows from results of Schmidt that $\widetilde{H}_\chi(n)$ has $\mathfrak{p}$-integral Fourier coefficients.

Proposition

Let e denote Hida's ordinary projection operator, acting on $S(N, \mathcal{O}) \otimes \mathbb{Q}$. Then $e\widetilde{H}_\chi(n)^{hol}$ has $\mathfrak{p}$-integral Fourier coefficients.

Congruences between L-functions

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- Putting everything together, we have that

$$(f, e\widetilde{H}_\chi(n)^{\mathrm{hol}})_N = \left(\frac{p^{n-1}}{\psi(p)\alpha_p^2}\right)^{m_\chi} \cdot \Gamma(n) \cdot G(\bar{\eta}) \cdot \frac{D_f(\chi, n)}{\pi^n \Omega}.$$

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- The functional $S_k(N, \mathcal{O}) \to \mathcal{O}$ given by $(\cdot, e\widetilde{H}_\chi(n)^{\mathrm{hol}})_N$ is $\mathcal{O}$-linear and continuous.
- Therefore, if f_1 and f_2 are $\mathfrak{p}$-congruent, it follows that

$$(f_1, e\widetilde{H}_\chi(n)^{\mathrm{hol}})_N \equiv (f_2, e\widetilde{H}_\chi(n)^{\mathrm{hol}})_N \mod \mathfrak{p}.$$

- We obtain the congruence

$$\begin{aligned}&\left(\frac{p^{n-1}}{\psi(p)\alpha_p^2}\right)^{m_\chi}\cdot\Gamma(n)\cdot G(\bar{\eta})\cdot\frac{D_{f_1}(\chi,n)}{\pi^n\Omega}\\ \equiv u&\left(\frac{p^{n-1}}{\psi(p)\alpha_p^2}\right)^{m_\chi}\cdot\Gamma(n)\cdot G(\bar{\eta})\cdot\frac{D_{f_2}(\chi,n)}{\pi^n\Omega}\mod \mathfrak{p}.\end{aligned}$$

Corollary

We have $L_p(\mathrm{Sym}^2(f_1)\otimes\psi)\equiv uL_p(\mathrm{Sym}^2(f_2)\otimes\psi) \pmod{\mathfrak{p}}$*, where* u *is a p-adic unit the congruence is that of elements in the completed group algebra* $\mathcal{O}[[\mathbb{Z}_p^\times]]$.

Thank you!