§ 1. Introduction: Kato classes.

$E/\mathbb{Q}$ elliptic curve

$p > 2$ good prime

$$V = T_pE \underset{\mathbb{Z}_p}{\otimes} \mathbb{Q}_p$$

<u>Kato</u>: $\exists\ z^{\mathrm{kato}} \in H^1(\mathbb{Q}, V)$ s.t.

$$\exp_p^*\left(\mathrm{res}_p(z^{\mathrm{kato}})\right) \doteq \frac{L(E,1)}{\Omega}$$

where

$$\exp_p^* : H^1(\mathbb{Q}_p, V) \xrightarrow{\exp_{BK}^*} \mathrm{Fil}^0 D_{dR}(V) = \mathbb{Q}_p\omega \to \mathbb{Q}_p.$$

<u>Conjecture 1</u> (Perrin-Riou, 1993).

Suppose $L(E,1) = 0$ $\left(\Rightarrow z^{\mathrm{kato}} \in \underset{\substack{\cap \\ E(\mathbb{Q})\otimes\mathbb{Q}_p}}{\mathrm{Sel}(\mathbb{Q}, V)}\right)$

If $\mathrm{Sel}(\mathbb{Q}, V) \neq \ker(\mathrm{res}_p)$,

then TFAE: (1) $z^{\mathrm{kato}} \neq 0$.

(2) $\dim_{\mathbb{Q}_p} \mathrm{Sel}(\mathbb{Q}, V) = 1$.

In that case, one expects

$$z^{\mathrm{Kato}} \doteq \log_p(P)\, P \in E(\mathbb{Q}) \otimes \mathbb{Q}_p$$

where P = Heegner points.

<u>Note</u>. Together with IMC, this predicts

$$\left.\begin{array}{c} L(E,1) = 0 \\ \& \\ \mathrm{sign}(E/\mathbb{Q}) = +1 \end{array}\right\} \Longrightarrow z^{\mathrm{Kato}} = 0.$$

§ 2. Main result

Suppose $p =$ ordinary.

Darmon–Rotger: depending on the choice of auxiliary $g \in S_1(N_g, \epsilon)$ with $L(1, V \otimes ad^0 V_g) \neq 0$

$\exists$ "generalised Kato class" $z^{DR} \in H^1(\mathbb{Q}, V)$.

s.t. $L(E,1) = 0 \Rightarrow z^{DR} \in Sel(\mathbb{Q}, V)$.

Conjecture 2 (Darmon–Rotger, 2016)

Suppose $L(E,1) = 0$ & $\mathrm{sign}(E/\mathbb{Q}) = +1$

$(\Rightarrow \mathrm{ord}_{s=1} L(E,s) \geqslant 2)$.

If $Sel(\mathbb{Q}, V) \neq \mathrm{Ker}(\mathrm{res}_p)$,

then TFAE: (1) $z^{DR} \neq 0$

(2) $\dim_{\mathbb{Q}_p} Sel(\mathbb{Q}, V) = 2$.

Let $K/\mathbb{Q}$ imaginary quadratic, $p = \mathfrak{p}\bar{\mathfrak{p}}$ split

Write

$$N_E = \underset{\substack{\uparrow \\ \text{split} \\ \text{in } K}}{N^+} \cdot \underset{\substack{\uparrow \\ \text{inert} \\ \text{in } K}}{N^-}$$

and suppose $N^- = \square$-free product of odd # of primes.

Theorem A (C.–Hsieh).

Suppose
- $E[p]$ abs. irreducible
- $E[p]$ ramified $\forall \ell \mid N^-$
- the auxiliary g has CM by K.

Then Conjecture 2 holds.

Remark. (i) If $g = \theta_\psi$,

then $L(s, V \otimes \mathrm{ad}^0 V_g)$

$$\begin{array}{c} L(s, V \otimes \mathrm{ad}^0 V_g) \\ \parallel \\ L(E^K, s) \cdot L(E/K, \chi, s) \end{array}$$

$$\chi = \psi/\psi^c$$

$\Longrightarrow$ can arrange $L(1, V \otimes \mathrm{ad}^0 V_g) \neq 0$

by Bump–Friedberg–Hoffstein & Vatsal.

(ii) The proof also shows

that if $\mathrm{rank}_{\mathbb{Z}} E(\mathbb{Q}) = 2$ & $\# Ш(E/\mathbb{Q})[p^\infty] < \infty$,

then

$$z^{DR} = \log_p(Q) P - \log_p(P) Q \pmod{\mathbb{Q}_p^\times},$$

where $(P, Q) = \mathbb{Q}$-basis of $E(\mathbb{Q}) \otimes \mathbb{Q}_p$.

§3. Construction of z^{DR}.

Let $f_E \in S_2(\Gamma_0(N_E)) \longleftrightarrow E$

$g = \theta_\psi$ with $L(1, V \otimes \mathrm{ad}^0 V_g) \neq 0$

$\alpha = \psi(\bar{\mathfrak{p}})$

Consider $\mathbb{g} = \{g_k\}_{k \geq 1}$ Hida family with $\underset{\alpha\text{-stab. of } \theta_\psi}{\overset{g_1}{\shortparallel}}$

$\mathbb{h} = \{h_k\}_{k \geq 1}$ Hida family with $\underset{\alpha^{-1}\text{-stab. of } \theta_{\psi^{-1}}}{\overset{h_1}{\shortparallel}}$

<u>Prop / Def</u> (Darmon – Rotger)

Let

$$z^{diag} := \lim_{\substack{k \to 1 \\ k \geq 2}} AJ_p(\Delta^{f_E g_k h_k})$$

$$\in H^1(\mathbb{Q}, V \otimes \underbrace{V_{g_1} \otimes V_{h_1}}_{\cong \mathbb{1} \oplus \mathrm{ad}^0 V_g})$$

and define

$$z^{DR} := \text{image of } z^{diag} \text{ in } H^1(\mathbb{Q}, V).$$

§4. Proof of $\mathrm{Sel}(\mathbb{Q}, V) \cong \mathbb{Q}_p^2 \Longrightarrow z^{DR} \neq 0$.

Use anticyclotomic Iwasawa theory

$$K_\infty = \bigcup_{n \geq 0} K_n \quad \text{anticycl } \mathbb{Z}_p\text{-ext'n of } K$$

$$\begin{tikzcd} K_\infty \arrow[d, dash, "\Gamma"] \\ K \end{tikzcd}$$

$$\Lambda := \mathbb{Z}_p[\![\Gamma]\!] \cong \mathbb{Z}_p[\![T]\!]$$

$$\gamma - 1 \longmapsto T$$

By construction,

$$z^{DR} = z_\infty\big|_{T=0}$$

for some $z_\infty \in H^1(K, V \otimes \Lambda)$

$$\cong H^1_{Iw}(K_\infty, V).$$

Proposition. $\exists$ Coleman power series map

$$\mathrm{Col} : H^1(\underset{\overset{\|}{K_\wp}}{\mathbb{Q}_p}, V \otimes \Lambda) \longrightarrow \Lambda \otimes \mathbb{Q}_p$$

s.t. $\boxed{\mathrm{Col}\left(\mathrm{res}_\wp(z_\infty)\right) \doteq \Theta_p^{BD}(T)}$ "ERL"

Bertolini–Darmon's Θ-elt.

interpolating $\sqrt{L^{alg}(E/K, \phi, 1)}$

$\phi : \Gamma \to \mu_{p^\infty}$.

Key to Thm. A : compute the leading term of ERL.

Note. Let $X := \mathrm{Sel}_{p^\infty}(E/K_\infty)^\wedge$

f.g. torsion, by Bertolini–Darmon & Vatsal Λ-module

In our setting, conjecture (after Mazur)

$$X \sim \left(\Lambda/J^2\right) \oplus \left(\Lambda/J^2\right) \oplus \left(\text{prime-to-}J\right)$$

where $J = (\gamma - 1) \subset \Lambda$ augmentation ideal.

$$\begin{array}{ccc} J = (\gamma-1) & \subset & \Lambda \\ \updownarrow & & \cong \\ (T) & \subset & \mathbb{Z}_p[\![T]\!] \end{array}$$

$\therefore$ the usual anticyclotomic p-adic reg $= 0$ in this case!

Howard: $\exists$ filtration

$$(\ast) \qquad \mathrm{Sel}(K,V) = S^{(1)} \supseteq S^{(2)} \supseteq \cdots \supseteq S^{(\infty)} = 0.$$

and derived p-adic heights

$$h_p^{(i)} : S^{(i)} \times S^{(i)} \longrightarrow \left(J^i/J^{i+1}\right) \otimes \mathbb{Q}_p$$

with $S^{(i+1)} =$ nullspace of $h_p^{(i)}$.

In our setting, $(\ast)$ reduces to

$$\underset{\substack{\uparrow \\ \mathrm{rk}=2}}{\mathrm{Sel}(K,V)} = S^{(1)} = S^{(2)} = \cdots = S^{(r)} \underset{\neq}{\supset} S^{(r+1)} = \cdots = 0.$$

for some $r \geq 2$.

$$\Rightarrow \quad X \sim \left(\Lambda/J^r\right) \oplus \left(\Lambda/J^r\right) \oplus \left(\begin{matrix}\text{prime}\\ \text{to } J\end{matrix}\right).$$

Set $\boxed{r_{an} := \mathrm{ord}_T\, \Theta_p^{BD}(T)}$

By ERL, we get $z^{DR} \in S^{(r_{an})}$.

By Skinner–Urban, $\left(\Theta_p^{BD}\right)^2 \supseteq \mathrm{char}_\Lambda(X)$

$$\Rightarrow r_{an} \leq r$$

$$\Rightarrow S^{(r_{an})} = \mathrm{Sel}(K,V). \left(= \mathrm{Sel}(\mathbb{Q},V)\right)$$

$\therefore$ The result follows from:

<u>Theorem B.</u> (C.–Hsieh)

For all $x \in S^{(r_{an})}$,

$$h_p^{(r_{an})}(z^{DR}, x) \doteq \underbrace{\Theta_p^{BD}(T)}_{\neq 0} \cdot \underbrace{\log_p(x)}_{\substack{\neq 0 \\ \text{on } \mathrm{Sel}(\mathbb{Q},V)}} \in \left(J^{r_{an}} / J^{r_{an}+1}\right) \otimes \mathbb{Q}_p$$

$\therefore$ $z^{DR} \neq 0$.