Rational points on algebraic curves in infinite towers of number fields

Anwesh Ray

University of British Columbia

anweshray@math.ubc.ca

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- Fix a number field K and let p be a prime number.
- We consider an infinite tower of number fields coming from cyclotomic extensions of K.
- For $n \geq 1$, let μ_{p^n} be the p^n-th roots of unity. Let $K(\mu_{p^\infty})$ be the infinite Galois extension of K generated by the p-power roots of unity

$$\mu_{p^\infty} = \cup_n \mu_{p^n}.$$

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- Thus, we have defined an infinite tower of number fields

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- Note that $\text{Gal}(K_n/K) \simeq \mathbb{Z}/p^n\mathbb{Z}$ and

  $$\text{Gal}(K_{\text{cyc}}/K) \simeq \varprojlim_n \text{Gal}(K_n/K) \simeq \varprojlim_n \mathbb{Z}/p^n\mathbb{Z} \simeq \mathbb{Z}_p.$$

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- When we need to emphasize the dependence on p, we use $K_n^{(p)}$ and $K_{\mathrm{cyc}}^{(p)}$.

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- We study a similar question for curves of higher genus.

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  2. Suppose that $\#X(K_n)$ is bounded as $n \to \infty$, let $m_0(p)$ be the minimal number such that $X(K_n) = X(K_{m_0(p)})$ for all $n > m_0(p)$. How can one better describe $m_0(p)$?

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  3. Under what conditions is $m_0(p) = 0$, i.e., under what conditions is $X(K) = X(K_{\text{cyc}})$?

Selmer groups

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- For each number field extension F/K, the Selmer group $\mathrm{Sel}_{p^\infty}(A/F)$ consists of Galois cohomology classes

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  satisfying suitable local conditions.
- It fits into a short exact sequence

$$0 \to A(F) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to \mathrm{Sel}_{p^\infty}(A/F) \to Ш(A/F)[p^\infty] \to 0.$$

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- Iwasawa introduced the completed algebra $\Lambda := \varprojlim_n \mathbb{Z}_p[\text{Gal}(K_n/K)] \simeq \mathbb{Z}_p[[T]]$. Here the formal variable T coincides with $\gamma - 1$, where γ is any choice of topological generator of $\text{Gal}(K_{\text{cyc}}/K)$.

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- The Pontryagin dual $\text{Sel}_{p^\infty}(A/K_{\text{cyc}})^\vee := \text{Hom}_{\text{cnts}}(\text{Sel}_{p^\infty}(A/K_{\text{cyc}}), \mathbb{Q}_p/\mathbb{Z}_p)$ is a finitely generated as a Λ-module.

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- It is not hard to show that the conjecture holds when $\mathrm{Sel}_{p^\infty}(A/K)$ is finite, i.e., when $\operatorname{rank} A(K) = 0$ and $\text{Ш}(A/K)[p^\infty]$ is finite.
- The λ-invariant of $\mathrm{Sel}_{p^\infty}(A/K_{\mathrm{cyc}})^\vee$ is given by

$$\lambda_p(A/K_{\mathrm{cyc}}) := \operatorname{rank}_{\mathbb{Z}_p}\left(\mathrm{Sel}_{p^\infty}(A/K_{\mathrm{cyc}})^\vee\right).$$

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Assume that $\mathrm{Sel}_{p^\infty}(A/K_{\mathrm{cyc}})^\vee$ *is a torsion* Λ*-module. There exists* $n_0 = n_0(p)$ *such that* $\operatorname{rank} A(K_n) = \operatorname{rank} A(K_{n_0})$ *for all* $n > n_0$*. Furthermore,* $\operatorname{rank} A(K_n)$ *is bounded above by* $\lambda_p(A/K_{\mathrm{cyc}})$*.*

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- This means in particular that if $\operatorname{rank} A(K) = \lambda_p(A/K_{\mathrm{cyc}})$, then, $\operatorname{rank} A(K) = \operatorname{rank} A(K_n)$ for all n.

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- We let $\alpha(p)$ denote the order of the torsion group.

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1. *A has good ordinary reduction at the primes above p,*
2. $\text{Sel}_{p^\infty}(A/K_{\text{cyc}})^\vee$ *is torsion over Λ.*

Then, $X(K_{\text{cyc}})$ is finite. Suppose that $X(K) \neq \emptyset$ and let m_0 be the minimum integer such that $X(K_n) = X(K_{m_0})$ for all $n > m_0$. Then, we have that

$$m_0 \leq n_0 + \lfloor \log_p \alpha(p) \rfloor.$$

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- Let $Q \in A(K_n)$ be a point such that $Q \notin A(K_{n-1})$.

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- This gives the bound $p^{n-n_0} \leq \alpha(p)$, or said differently,

$$n \leq n_0 + \lfloor \log_p \alpha(p) \rfloor.$$

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- Let $\widehat{\rho} : \mathrm{Gal}(\bar{K}/K) \to \mathrm{GSp}_{2g}(\widehat{\mathbb{Z}})$ be the Galois representation on $\mathrm{T}(A)$ ($\widehat{\mathbb{Z}} = \varprojlim \mathbb{Z}/n\mathbb{Z} = \prod_\ell \mathbb{Z}_\ell$).

- The representation $\widehat{\rho}$ is said to have *big image* if $(\text{image}\,\widehat{\rho}) \cap \mathrm{Sp}_{2g}(\widehat{\mathbb{Z}})$ is a finite index subgroup of $\mathrm{Sp}_{2g}(\widehat{\mathbb{Z}})$.

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Theorem (Serre, Pink)

Let $A_{/\mathbb{Q}}$ be an abelian variety and assume that $\mathrm{End}(A_{/\bar{\mathbb{Q}}}) = \mathbb{Z}$. Then the image of $\widehat{\rho}$ contains a finite index subgroup of $\mathrm{GSp}_{2g}(\widehat{\mathbb{Z}})$ provided $g = 1, 2$, or $g \geq 3$ is not in the set

$$\left\{\frac{1}{2}(2n)^k \mid n > 0, k \geq 3 \text{ is odd}\right\} \cup \left\{\frac{1}{2}\binom{2n}{n} \mid n \geq 3 \text{ is odd}\right\}.$$

A generalization of Imai's theorem

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Theorem

Let A be an abelian variety defined over K such that the image of $\widehat{\rho}$ is large. Let K_∞ be any pro-p extension of K. Furthermore assume that $A(K)$ has no p-torsion. Then, the torsion subgroup of $A(K_\infty)$ is finite.

- For any prime ℓ, let $\bar{G}_\ell$ be the image of the mod-ℓ representation

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- Then, if p is a prime such that $A(K)[p] = 0$, and K_∞ is any pro-p extension of K, then

$$A(K_\infty)_{\mathrm{tors}} \subseteq A(L)_{\mathrm{tors}},$$

where $L = K(A[\prod_{\ell\in\Sigma} \ell])$.

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1. *A has good ordinary reduction at the primes above p,*
2. *$\mathrm{Sel}_{p^\infty}(A/K_{\mathrm{cyc}})^\vee$ is torsion over Λ,*

$$X(K_{\mathrm{cyc}}^{(p)}) = X(K_{n_0(p)}^{(p)}).$$

Here we recall that $n_0(p)$ is the minimal value such that $\mathrm{rank}\, A(K_n) = \mathrm{rank}\, A(K_{n_0(p)})$.

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- Consider the power series expansion of $f(T)$

$$f(T) = a_0 + a_1 T + a_2 T^2 + \dots.$$

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- On the other hand, there is a p-adic anologue of the BSD formula for the leading coefficient

$$a_0 \sim \frac{\#\text{Ш}(A/K)[p^\infty] \times \prod_{v\nmid p} c_v^{(p)}(A/K) \times \left(\prod_{v|p} \#A(k_v)[p^\infty]\right)^2}{(\#A(K)[p^\infty])^2}.$$

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- A prime p is *anomalous* if $p \mid \#A(k_v)$ for some prime $v|p$.

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Theorem

Let $A_{/K}$ be an abelian variety for which $\widehat{\rho}$ has big image. Then, 100% of primes p are non-anomalous.

Thank you!